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SENSITIVITY ANALYSIS AND OPTIMIZATION OF AEROELASTIC STABILITY

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Abstract—The present paper deals with problems concerning sensitivity analysis and optimization of aeroelastic stability of distributed systems. In Sections 1–3 the optimization problem of aeroelastic stability of a slender wing in incompressible flow is formulated and solved. The optimal control function $h^*(y)$ to be determined represents the mass and stiffness distribution along a wing span. Sensitivity analysis of flutter systems is developed. First, the gradients of flutter and divergence critical speeds are derived, and necessary optimality conditions are obtained. Then the solution technique is described, and numerical results are presented. In Section 4 the problem of determining the optimal distribution of nonstructural mass along the wing span, is considered. The optimality conditions are established and the bang–bang optimal distributions are obtained.

INTRODUCTION

In recent years problems in structural optimization taking aeroelastic constraints into account have been studied both from a theoretical point of view and with regard to their applications. Many papers are devoted to the optimal problem of aeroelastic stability[1–18]. The formulation of necessary conditions and their application in numerical procedures are connected with considerable difficulties, and in spite of many papers in this field, these problems have not yet been overcome but have given rise to different, often dubious, results. This applies to discrete as well as to continuous systems.

The search for optimal structures is intimately connected with a sensitivity analysis of the structure with respect to all the design variables determining the aeroelastic behaviour of the structure. Sensitivity analysis itself provides the designer with some important information and indicates ways of improving the structure in a rational manner[19–21].

In the present paper problems concerning sensitivity analysis and optimization of continuous structures with respect to aeroelastic stability are considered. These questions are studied in Sections 1–4, where the problem of bending-torsional flutter of a wing is considered and solved numerically.

1. BASIC RELATIONS

Let us consider vibrations of a long and thin wing in an incompressible air flow. We assume that the wing may be treated as a slender elastic beam with a straight elastic axis Oy, which is perpendicular to the centerline of the fuselage, see Fig. 1. The vertical axis Oz is directed upwards, perpendicular to the plane of the figure. The inertia axis is by a solid line indicated in Fig. 1.

The deformation of the wing is characterized by the deflection $w(y, \tau)$ and the angle of twist $\theta(y, \tau)$ about the elastic axis. In terms of these quantities, the equations of motion of the wing take the form[22, 23]

$$\frac{\partial^2}{\partial y^2}\left(EI\frac{\partial^2 w}{\partial y^2}\right) + m\frac{\partial^2 w}{\partial \tau^2} - m\sigma\frac{\partial^2 \theta}{\partial \tau^2} = L_a$$

$$-\frac{\partial}{\partial y}\left(GJ\frac{\partial \theta}{\partial y}\right) - m\sigma\frac{\partial^2 w}{\partial \tau^2} + I_m\frac{\partial^2 \theta}{\partial \tau^2} = M_a\,. \tag{1.1}$$

Here, EI and GJ denote the bending and torsional rigidities, m and I_m are the mass and the moment of inertia per unit length, σ is the distance of the axis of inertia from the elastic axis, and L_a and M_a are the aerodynamic lift force and the pitching moment, respectively, about the elastic axis.

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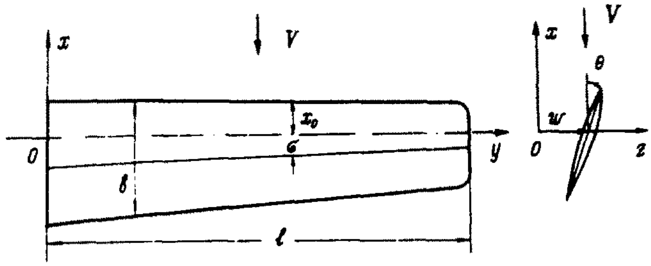

Fig. 1.

To describe the aerodynamic loads we use the so-called strip theory and assume a condition of quasistationarity, according to which the aerodynamic characteristics of a wing in unsteady motion at any time and at any strip are characterized by a rigid wing moving at constant velocity and constant angular velocity, equal to the instant velocities of the strip [22, 23]. The aerodynamic forces are then given by the relations

$$L_a = c_y{}^a \rho b V^2\left[\theta + \frac{b}{V}\left(\frac{3}{4} - \frac{x_0}{b}\right)\frac{\partial\theta}{\partial\tau} - \frac{1}{V}\frac{\partial w}{\partial\tau}\right]$$

$$M_a = c_m{}^a \rho b^2 V^2\left[\theta + \frac{b}{V}\left(\frac{3}{4} - \frac{x_0}{b} - \frac{\pi}{16 c_m{}^a}\right)\frac{\partial\theta}{\partial\tau} - \frac{1}{V}\frac{\partial w}{\partial\tau}\right] \qquad (1.2)$$

where b is the chord of the wing and x_0 the distance between the leading edge of the wing and its elastic axis, and where ρ and V are the density and the speed of the flow, respectively. The theoretical values of the aerodynamic coefficients $c_y{}^a$ and $c_m{}^a$ for a thin wing of infinite span are given by $c_y{}^a = \pi$ and $c_m{}^a = \pi(x_0/b - 1/4)$, respectively.

The boundary conditions for a cantilever beam are given by the relations

$$w = \frac{\partial w}{\partial y} = \theta = 0 \text{ at } y = 0$$

$$EI\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(EI\frac{\partial^2 w}{\partial y^2}\right) = GJ\frac{\partial\theta}{\partial y} = 0 \text{ at } y = l. \qquad (1.3)$$

The system of eqns (1.)–(1.3) represents a linear and homogeneous eigenvalue problem. The solution is found in the general form

$$w(y, \tau) = u(y)e^{\nu\tau}, \quad \theta(y, \tau) = v(y)e^{\nu\tau} \qquad (1.4)$$

where ν is an eigenvalue and where $u(y)$ and $v(y)$ are the corresponding eigenfunctions. Due to the fact that the forces are non-conservative, the eigenvalue ν is generally a complex quantity $\nu = q + i\omega$; the eigenfunctions u and v are therefore also complex quantities, $u = u_1 + iu_2$ and $v = v_1 + iv_2$, respectively.

Depending upon the speed V of the flow, the amplitude of the vibrations can decrease (Re $\nu < 0$, stability) or increase (Re $\nu > 0$, instability). Two types of instability may be distinguished: dynamic (flutter) and static (divergence) [24]. The critical flutter speed is characterized by the values Re $\nu = 0$, Im $\nu = \omega \neq 0$, where ω is the frequency of flutter. The speed of critical divergence is determined by $\nu = 0$.

Let us write the equations of motion of the wing at the point of flutter. For this purpose we substitute (1.4) into the eqns (1.1)–(1.3) with $\nu = i\omega$. Taking $V = V_f$ we get a system of equations for the eigenfunctions $u(y)$ and $v(y)$ in the form

$$Lf \equiv \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad (1.5)$$

where L_{ij} are linear differential operators of the form

$$L_{11} = \frac{d^2}{dy^2}\left(EI\frac{d^2}{dy^2}\right) - m\omega^2 + i\omega c_y{}^a\rho b V_f$$

$$L_{12} = m\sigma\omega^2 - c_y{}^a\rho b V_f^2 - i\omega c_y{}^a\rho b^2 V_f\left(\frac{3}{4} - \frac{x_0}{b}\right)$$

$$L_{21} = m\sigma\omega^2 + i\omega c_m{}^a\rho b^2 V_f \tag{1.6}$$

$$L_{22} = -\frac{d}{dy}\left(GJ\frac{d}{dy}\right) - I_m\omega^2 - c_m{}^a\rho b^2 V_f^2 - i\omega c_m{}^a\rho b^3 V_f\left(\frac{3}{4} - \frac{x_0}{b} - \frac{\pi}{16c_m{}^a}\right).$$

The boundary conditions for u and v follow from (1.3)

$$u = \frac{du}{dy} = v = 0 \text{ at } y = 0$$

$$EI\frac{d^2u}{dy^2} = \frac{d}{dy}\left(EI\frac{d^2u}{dy^2}\right) = GJ\frac{dv}{dy} = 0 \text{ at } y = l. \tag{1.7}$$

We will now consider the problem of divergence. For this purpose, we put $\nu = 0$ in (1.4)–(1.6), whereby we get the self-adjoint, positive-definite eigenvalue problem [24]

$$\frac{d}{dy}\left(GJ\frac{dv_d}{dy}\right) + c_m{}^a\rho b^2 V_d^2 v_d = 0$$

$$v_d(0) = 0, \left(GJ\frac{dv_d}{dy}\right)_{y=l} = 0. \tag{1.8}$$

Here, $v_d(y)$ denotes the eigenfunction at divergence, and the critical divergence speed V_d is determined by the lowest eigenvalue of the problem (1.8).

The relations (1.5)–(1.8) may be rewritten in non-dimensional form using the non-dimensional quantities

$$\bar{y} = y/l,\ \bar{\sigma} = \sigma/l,\ \bar{x}_0 = x_0/l,\ \bar{b} = b/l,\ \bar{\rho} = \rho l^2/\tilde{m}$$

$$\bar{m} = m/\tilde{m},\ \bar{I}_m = I_m/\tilde{m}l^2,\ \bar{EI} = EI/\tilde{EI},\ \bar{GJ} = GJ/\tilde{EI} \tag{1.9}$$

$$\bar{u} = u/l,\ \bar{V} = Vl\sqrt{\left(\frac{\tilde{m}}{\tilde{EI}}\right)},\ \bar{\omega} = \omega l^2\sqrt{\left(\frac{\tilde{m}}{\tilde{EI}}\right)}$$

where $\tilde{m}$ and $\tilde{EI}$ represent some mean values of mass density and bending rigidity, respectively.

Note that the relations (1.5)–(1.8) remain unchanged in the new variables, except that, in the boundary conditions, we have to put 1 instead of l. From now on we shall use non-dimensional quantities and drop the bars over the symbols.

Let us now introduce the control function $h(y)$. We assume that the cross-section of the wing is a thin-walled closed profile of arbitrary geometry. If the thickness is multiplied by h, the rigidities, the mass and the inertia of any cross section must also be multiplied by h, while σ and x_0 remain unchanged. We shall therefore write

$$EI(y) = EI_0(y)h(y),\ GJ(y) = GJ_0(y)h(y)$$

$$I_m(y) = I_{m_0}(y)h(y),\ m(y) = m_0(y)h(y) \tag{1.10}$$

where EI_0, GJ_0, I_{m_0}, and m_0 are some fixed stiffness and mass functions and where $h(y)$ serves as a non-dimensional control function. For physical reasons, we must assume that $h(y) \geq 0$. A variation of this function will lead to new distributions of masses and stiffnesses and will therefore influence the critical flutter and divergence speeds.

Our first aim is to determine the influence of a small variation of the control function on the critical flutter and divergence speed. Our second aim is to raise the critical speed by means of suitable variations of the control function $h(y)$, keeping the total mass of the wing unchanged.

2. FUNCTIONAL GRADIENTS OF CRITICAL FLUTTER AND DIVERGENCE SPEEDS

In this section we obtain the increment of the critical flutter and divergence speeds due to a variation δh of the control function.

For this purpose we introduce the adjoint flutter problem [25–27] to the problem (1.5), (1.7):

$$L^T p \equiv \begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0. \tag{2.1}$$

The operators L_{ij} are defined by the expressions (1.6). The functions $\phi(y)$ and $\psi(y)$ are complex and of the form $\phi = \phi_1 + i\phi_2$, $\psi = \psi_1 + i\psi_2$. The boundary conditions have the form

$$\phi = \frac{d\phi}{dy} = \psi = 0 \text{ at } y = 0$$

$$EI\frac{d^2\phi}{dy^2} = \frac{d}{dy}\left(EI\frac{d^2\phi}{dy^2}\right) = GJ\frac{d\psi}{dy} = 0 \text{ at } y = 1. \tag{2.2}$$

It can be shown that the critical speed of flutter and frequency of this problem (2.1)–(2.2) coincide with those of the problem (1.5)–(1.7) because the problems are adjoint. These problems are linear and homogeneous with respect to the vector-functions f and p. Hence, any solution is only determined up to an arbitrary complex multiplier.

We proceed now to compute the variations. Consider first the main flutter problems (1.5), (1.7) taking relations (1.10) into account. An increment $\delta h(y)$ leads to increments of V_f, ω, and the complex vector function $f(y)$. The increments $\delta h(y)$, δV_f, and $\delta\omega$ are real quantities, and the variation of the complex function f has the form

$$\delta f(y) = \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \begin{pmatrix} \delta u_1 + i\delta u_2 \\ \delta v_1 + i\delta v_2 \end{pmatrix}.$$

Now for the problems (1.5) and (1.7), we write the equation in variations

$$K(\delta h)f + L_{V_f} f\delta V_f + L_\omega f\delta\omega + L\delta f = 0 \tag{2.3}$$

$$\delta u = \frac{d\delta u}{dy} = \delta v = 0 \text{ at } y = 0$$

$$EI_0\delta h\frac{d^2u}{dy^2} + EI\frac{d^2\delta u}{dy^2} = \frac{d}{dy}\left(EI_0\delta h\frac{d^2u}{dy^2} + EI\frac{d^2\delta u}{dy^2}\right) = 0 \tag{2.4}$$

$$GJ_0\delta h\frac{dv}{dy} + GJ\frac{d\delta v}{dy} = 0 \text{ at } y = 1$$

where the matrices L_{V_f} and L_ω are produced by the matrix L, see (1.5) and (1.6), by formal differentiation with respect to the variables V_f and ω. The matrix operator $K(\delta h)$ is given by the expression

$$K(\delta h) = \begin{bmatrix} \dfrac{d^2}{dy^2}\left(EI_0\delta h\dfrac{d^2}{dy^2}\right) - m_0\omega^2\delta h & m_0\sigma\omega^2\delta h \\ m_0\sigma\omega^2\delta h & -\dfrac{d}{dy}\left(GJ_0\delta h\dfrac{d}{dy}\right) - I_{m_0}\omega^2\delta h \end{bmatrix}.$$

Further, we multiply eqn (2.3) by the vector function $p^T(y) = (\phi(y), \psi(y))$, where p is the solution of the adjoint flutter problem (2.1) and (2.2), and integrate between 0 and 1

$$\int_0^1 [p^T K(\delta h) f + (p^T L_{V_f} f)\delta V_f + (p^T L_\omega f)\delta\omega + p^T L\delta f]\mathrm{d}y = 0. \tag{2.5}$$

Integrating by parts, taking the boundary conditions (2.2), (2.4) into account, we find that the last term of the integrand in (2.5) vanishes

$$\int_0^1 p^T L\delta f \mathrm{d}y = \int_0^1 \delta f^T L^T p \mathrm{d}y = 0.$$

Here, the last equality follows from (2.1). The first term in (2.5) can be rewritten into the form

$$\int_0^1 p^T K(\delta h) f \mathrm{d}y = \int_0^1 H\delta h \mathrm{d}y$$

$$H = EI_0 \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}\frac{\mathrm{d}^2 \phi}{\mathrm{d}y^2} + GJ_0 \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}\psi}{y} + \omega^2 p^T \begin{bmatrix} -m_0 & m_0\sigma \\ m_0\sigma & -I_{m_0} \end{bmatrix} f. \tag{2.6}$$

Using the notation

$$A = \int_0^1 (p^T L_{V_f} f)\mathrm{d}y, \; B = \int_0^1 (p^T L_\omega f)\mathrm{d}y \tag{2.7}$$

the equation (2.5) gets the form

$$\int_0^1 H\delta h \mathrm{d}y + A\delta V_f + B\delta\omega = 0. \tag{2.8}$$

Note that the function H is a complex function of the real variable y and that the constants A and B are complex. Let us multiply (2.8) by $\bar{B}$, the complex conjugate to B, and separate the imaginary part. Because $Im(B\bar{B}) = 0$ and δV_f, and $\delta\omega$ are both real, we get the following expression for the variation

$$\delta V_f = \int_0^1 g\delta h \mathrm{d}y, \; g = -\frac{Im(H\bar{B})}{Im(A\bar{B})} \tag{2.9}$$

It follows that the function g is the gradient of the functional for the critical flutter speed with respect to the control function h.

The variation of the flutter frequency can be obtained from (2.8) in a similar manner,

$$\delta\omega = \int_0^1 t\delta h \mathrm{d}y, \; t = -\frac{Im(H\bar{A})}{Im(B\bar{A})}. \tag{2.10}$$

Thus, in order to calculate the gradients g and t, it is necessary to solve the main and the adjoint flutter problems (1.5), (1.7) and (2.1), (2.2), and to determine the complex vector functions $f(y)$ and $p(y)$ and the real quantities V_f and ω. From (2.6) and (2.7) we can then obtain the complex constants A and B and the complex function H, and hence determine the gradients g and t by means of the relations (2.9) and (2.10). Note that the eigenfunctions f and p are only determined up to arbitrary complex multipliers because the main and adjoint flutter problems are both homogeneous, but that the gradients g and t remain unchanged if f and p are multiplied by arbitrary complex constants.

We now proceed to determining the gradient of the critical divergence speed with respect to the control function $h(y)$. Since the divergence problem (1.8) is self-adjoint and positive-definite, Rayleigh's minimum principle is valid [28], and we have

$$V_d^2 = \min_v \frac{\int_0^1 GJ\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 \mathrm{d}y}{\int_0^1 c_m{}^a \rho b^2 v^2 \mathrm{d}y}. \tag{2.11}$$

Here, the function v must satisfy the kinematic boundary condition $v(0) = 0$ and be continuously differentiable. Variation of (2.11) readily gives

$$\delta V_d = \int_0^1 e\delta h \mathrm{d}y,\ e = \frac{GJ_0\left(\frac{\mathrm{d}v_d}{\mathrm{d}y}\right)^2}{2V_d \int_0^1 c_m{}^a \rho b^2 v_d^2 \mathrm{d}y}. \tag{2.12}$$

Note that because the divergence problem is self-adjoint, it is not necessary to introduce an adjoint problem for determination of the gradient e.

The method described above for determining the gradients of the critical speeds may also be used to obtain gradients with respect to some other independent functions or parameters of the problem. For example, the derivative of the critical flutter speed with respect to the density ρ of the flow is given by the formula

$$\frac{\partial V_f}{\partial \rho} = -\frac{Im(C\bar{B})}{Im(A\bar{B})}.$$

Here, the quantities A and B are defined in eqn (2.7), and C is given by

$$C = \int_0^1 (p^T L_\rho f) \mathrm{d}y$$

where the matrix L_ρ is obtained by differentiating L with respect to ρ.

Knowing the gradients of the critical flutter and divergence speeds with respect to, e.g. some mass distributions or other parameters, we can improve the characteristics of aeroelastic stability for a structure in a rational manner.

In fact, the influence of certain parameters on the characteristics of dynamic stability has been investigated by several researchers, see, e.g. [22–24, 29–34]. In many dynamic stability problems, the methods of similarity and dimensions [22, 30] provide the main basis for the structural analysis, but generally speaking, analysis of the influence of different parameters on the region of stability is quite difficult, because the critical values cannot normally be expressed implicitly in terms of essential parameters of the system. In Sections 2 and 4 of the present paper, a derivation is presented of the relations that describe the sensitivity of critical speeds with respect to distributed and discrete parameters, which greatly influence the aeroelastic behaviour of the system. In this derivation, use is made of the so-called adjoint system.

The method of sensitivity analysis developed in this paper becomes most effective when a large finite (or infinite) number of essential aeroelastic parameters are considered, because the calculation of the gradient of the critical flutter speed only call for the main and the adjoint flutter problems to be solved once. In contrast to this, a gradient calculation based on numerical differentiation of the critical flutter (or divergence) speed would require the flutter (or divergence) problem to be solved $(N+1)$ times if N essential aeroelastic parameters are taken into account. The method suggested in this paper may also prove useful for analyses of static and dynamic instability phenomena for other distributed and discrete systems.

Sensitivity of critical values of stability with respect to vanishing internal structural damping [31–34] should be mentioned as an example that would demand special analysis.

3. OPTIMIZATION OF AEROELASTIC STABILITY

We will now consider the problem of maximizing the critical speed at which stability is lost, assuming the total mass of structure material to be given. Mathematically, this problem may be posed as

$$\max_{h\in\Omega} \min [V_f(h), V_d(h)] = \min [V_f(h^*), V_d(h^*)]$$

$$\Omega = \left\{ h(y) : M(h) = \int_0^1 h(y) m_0(y) \mathrm{d}y = M_0 \right\} \tag{3.1}$$

i.e. as a problem of determining the mass distribution $h^*(y)$, which, within the constraint of given total mass $M(h) = M_0$, maximizes the smaller of the critical flutter and the critical divergence speeds.

Let us first estimate the highest value of the critical speed that causes loss of stability. To this end, we consider the problem of maximizing the critical divergence speed for a given volume of material. The solution to this problem is well known[2, 3, 14]

$$h_d^0 = \frac{c_m{}^a \rho V_d^{02}}{\sqrt{(GJ_0 m_0)}} \int_y^1 \left(\int_0^y \sqrt{\left[\frac{m_0}{GJ_0}\right]} \mathrm{d}y \right) b^2 \mathrm{d}y, \; v_d{}^0 = \int_0^y \sqrt{\left(\frac{m_0}{GJ_0}\right)} \mathrm{d}y$$

where the maximum divergence speed V_d^0 is given by the expression

$$V_d^0 = \left(\int_0^1 m_0 \mathrm{d}y \right)^{1/2} \left[c_m{}^a \rho \int_0^1 \left(b \int_0^y \sqrt{\left[\frac{m_0}{GJ_0}\right]} \mathrm{d}y \right)^2 \mathrm{d}y \right]^{-1/2} \tag{3.2}$$

Using the estimate [35]

$$\max_{h\in\Omega} \min (V_f, V_d) \leq \max_{h\in\Omega} V_d = V_d^0$$

which is valid due to the inequality $\min(V_d, V_f) \leq V_d$, we find

$$\max_{h\in\Omega} \min (V_f, V_d) \leq V_d^0 \tag{3.3}$$

where V_d^0 is defined by (3.2).

We now proceed to deriving the necessary conditions of optimality for the problem in (3.1). The variation of the functional of the total mass is

$$\delta M = \int_0^1 m_0(y) \delta h(y) \mathrm{d}y$$

so the gradient of the mass functional is simply given by $m_0(y)$. Taking into account the gradients g and e obtained in the previous section and using the results of a maximin approach, see [36], we obtain the necessary optimality conditions for the optimal mass distribution $h^*(y)$,

$$\begin{gathered} \lambda e(y) + (1-\lambda) g(y) + \mu m_0(y) = 0 \\ \lambda = 0 \text{ if } V_f(h^*) < V_d(h^*) \\ \lambda = 1 \text{ if } V_d(h^*) < V_f(h^*) \\ 0 \leq \lambda \leq 1 \text{ if } V_d(h^*) = V_f(h^*). \end{gathered} \tag{3.4}$$

Here, the multipliers λ and μ are defined by the isoperimetric conditions of the problem.

In the numerical solution of the maximum problem (3.1) it is necessary to vary the initial distribution $h(y)$ in order to increase the smaller critical speed for loss of stability. Assuming that the initial distribution $h(y)$ satisfies the condition $M(h) = M_0$, we may take an improved design variation in the form

$$\delta h(y) = \alpha(y)[\lambda e(y) + (1-\lambda)g(y) + \mu m_0(y)] \tag{3.5}$$

where $\alpha(y)$ is a so-called "gradient step", that is, an arbitrary positive function chosen by the researcher.

The two as yet unknown multipliers λ and μ are defined by the isoperimetric conditions. We will first consider the case $V_f(h) < V_d(h)$. Taking $\lambda = 0$ and defining μ via substituting (3.5) into the condition $\delta M = 0$, we obtain

$$\lambda = 0,\ \mu = -\int_0^1 \alpha g m_0 \mathrm{d}y \Big/ \int_0^1 \alpha m_0^2 \mathrm{d}y. \tag{3.6}$$

Analogously, we obtain for the case of $V_d < V_f$,

$$\lambda = 1,\ \mu = -\int_0^1 \alpha e m_0 \mathrm{d}y \Big/ \int_0^1 \alpha m_0^2 \mathrm{d}y. \tag{3.7}$$

Finally, if we have $V_f = V_d$, we can substitute (3.5) into the conditions $\delta V_f = \delta V_d$ and $\delta M = 0$, and obtain the following system of linear equations

$$\lambda \int_0^1 \alpha(e-g)^2 \mathrm{d}y + \mu \int_0^1 \alpha(e-g)m_0 \mathrm{d}y = -\int_0^1 \alpha(e-g)g \mathrm{d}y$$

$$\lambda \int_0^1 \alpha(e-g)m_0 \mathrm{d}y + \mu \int_0^1 \alpha m_0^2 \mathrm{d}y = -\int_0^1 \alpha g m_0 \mathrm{d}y. \tag{3.8}$$

It is easily seen that the determinant of this system is a Gram determinant[19]. It is equal to zero only when $e-g$ and m_0 are linearly dependent functions.

In the case of $V_f = V_d$ it can be shown that the variations δV_f and δV_d can be expressed by

$$\delta V_f = \delta V_d = \int_0^1 \alpha[\lambda e + (1-\lambda)g + \mu m_0]^2 \mathrm{d}y \geq 0$$

which implies that our algorithm meets the condition $M(h) = M_0$ and increases the critical speed of instability at each step of variation. The same holds good for the cases of $V_f < V_d$ and $V_d < V_f$.

In fact, the solution procedure is similar to that developed by Niordson[19]. The main and the adjoint flutter problem must be solved at each step of the gradient procedure for computing the improved mass distribution. Solution of the main flutter problem (1.5), (1.7) is performed by the method of successive iterations as described in [22], and the adjoint flutter problem is solved by the same method with only insignificant differences in the computer program. A comparison of the values obtained for V_f and ω in these two problems yields an effective check of the accuracy of the computations.

Also the divergence problem (1.8) is to be solved at each step of the gradient procedure. The solution of this problem is performed by the method of successive iterations described in [28] and gives us the critical divergence speed V_d, the eigenfunctions $v_d(y)$ and the gradient function $e(y)$.

The numerical procedure (with n designating the iteration number) consists of the following iteration steps

(1) For the first iteration ($n = 1$), assume a distribution $h^{(n)}(y)$ that satisfies the constraint $M(h^{(n)}) = M_0$. For $n > 1$, apply the distribution obtained by the end of the previous iteration.

(2) Solve the main and adjoint flutter problems (1.5), (1.7) and (2.1), (2.2), respectively, and obtain the eigenfunctions $u^{(n)}$, $v^{(n)}$, $\phi^{(n)}$, $\psi^{(n)}$ and their derivatives, together with the values $V_f^{(n)}$ and $\omega^{(n)}$.

(3) Determine $H^{(n)}(y)$, $A^{(n)}$ and $B^{(n)}$ by means of (2.6) and (2.7), and apply (2.9) to obtain the gradient $g^{(n)}$.

(4) Solve the divergence problem (1.8) to obtain the critical quantity V_d, along with the eigenfunction v_d and its first derivative. Compute the gradient function $e^{(n)}$ by means of (2.12).

(5) Compare the critical values $V_f^{(n)}$ and $V_d^{(n)}$, and determine the constants $\lambda^{(n)}$ and $\mu^{(n)}$ by means of the relations (3.6)–(3.8).

(6) Determine the variation $\delta h^{(n)}$ by (3.5) and compute the distribution function for the subsequent iteration as $h^{(n+1)} = h^{(n)} + \delta h^{(n)}$.

As an numerical example, let us consider a rectangular wing with uniform initial distributions EI_0, GJ_0, I_{m_0}, m_0, b, l, σ and x_0. These parameters, together with the quantities c_y^a and c_m^a, are taken to be equal to those in Grossman[22], wing No. 3. The mean quantities $\bar{m}$ and $\bar{EI}$ in (1.9) are chosen to be equal to m_0 and EI_0, respectively. Note that according to (1.9), (1.10), six nondimensional parameters and the function $h(y)$,

$$\bar{\sigma} = \sigma/l,\ \bar{x}_0 = x_0/l,\ \bar{b} = b/l,\ \bar{\rho} = \rho l^2/m_0$$

$$\bar{I}_m = (I_{m_0}/m_0 l^2)h(y),\ \bar{GJ} = (GJ_0/EI_0)h(y),\ \bar{m} = \bar{EI} = h(y),$$

are required, besides the coefficients c_y^a and c_m^a, for solution of the main and adjoint flutter problems. Due to the constant m_0, the isoperimetric condition $M(h) = M_0$ in (3.1) takes the form $\int_0^1 h(y)\mathrm{d}y = 1$. In the numerical calculations the interval [0, 1] was divided into $N = 20, 40$ equal subintervals, and numerical integration was performed in applying the trapezoidal rule.

We consider first the uniform initial distribution $h^{(0)} = (y) = 1$, which corresponds to uniform mass and stiffness distributions along the wing span. For this distribution, the critical divergence speed is much greater than the flutter speed, $V_f = 29.4 < V_d = 59.4$. Hence, the optimization algorithm described in the foregoing is reduced to maximization of flutter speed at fixed total mass, see (3.5) and (3.6).

Figure 2 shows how the distribution $h(y)$ and the gradient $g(y)$ of the flutter speed develop during iterations. It is clearly indicated that the distribution which corresponds to a local maximum of the critical flutter speed tends to infinity when $y \to 1$. Physically, this means that a concentrated mass should be located at the free tip of the wing in order to increase the critical speed. For a correct statement of the problem, an upper constraint $h(y) \leq h_{max}$ should therefore

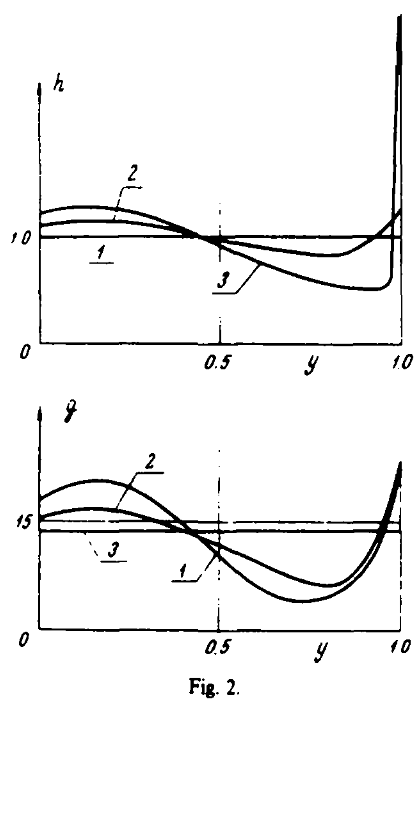

Fig. 2.

be specified for the distribution function. In the present case we take $h_{max} = 4.5$, and the function $h_0(y)$ and the appropriate sensitivity function $g_0(y)$ that correspond to the local maximum of the critical flutter speed are indicated by the number 3 in Fig. 2.

During the iteration process the critical flutter speed increases from the value $V_f = 29.4$, which corresponds to the initial distribution $h^{(0)}(y) = 1$, and up to the value $V_f = 30.9$ for the distribution $h_0(y)$. At the same time, the flutter frequency changes from the initial value 107.1 and up to 112.4. Thus, the increase of the flutter speed is rather small: only about 5 pct. The critical divergence speeds for the distributions $h(y)$ shown in Fig. 2 are much greater than the flutter speeds.

The eigenfunctions $u(y) = u_1(y) + iu_2(y)$ and $v(y) = v_1(y) + iv_2(y)$ associated with the distribution $h_0(y)$ are illustrated in Fig. 3. These functions are defined up to an arbitrary complex multiplier, and we used normalization condition $u(1) = 2$ in this example. Note that the character of these curves does not change during the iteration process.

The above distribution $h_0(y)$ is obtained not only if we start the iterations from $h^{(0)}(y) = 1$, but also if we use other initial functions such as, e.g. $h^{(0)}(y) = 1.95 - 1.9y$.

However, it turns out that the distribution $h_0(y)$ only corresponds to a local maximum of the critical flutter speed at given total mass. Thus, starting the iteration process (3.5), (3.6) from the function $h_1(y) = 2.7(1 - y)^2 + 0.1$ leads to other results. The distribution $h_1(y)$ indicated by number 1 in Fig. 4 is characterized by a rather low value of the flutter speed $V_f = 29.1$ ($V_d = 57.5$), but for this distribution, the absolute value of the gradient $g_1(y)$ of the functional turns out to be large (Fig. 4). Note also that the function $g_1(y)$ in Fig. 4 differs from that presented in Fig. 2 by changing its sign. From Fig. 4 we see that the region close to the free tip of the wing ($y \to 1$) is very sensitive to variations of $h(y)$. A small removal of material from this region will lead to a rapid increase in the value of V_f (thin tip effect).

Since the gradient $g_1(y)$ attains negative values in the region [0, 1], we may conclude that V_f can be increased if we reduce the total mass of the wing. Therefore, some distributions $h(y)$ must exist for which a reduction of mass does not contradict an increase of the critical flutter speed.

A similar effect of gradient function attaining negative values is emphasized in the recent work of Pedersen[21], where problems of maximizing the frequency of free vibrations for beams are considered. Note that according to (2.12), the gradient of the divergence speed is always positive, so an increase in the amount of material will always lead to an increase of the divergence speed.

A few iterations (3.5), (3.6) started from the distribution $h_1(y)$ lead to an essential increase in the critical flutter speed. The distribution $h_2(y)$ indicated by the number 2 in Fig. 4 is associated

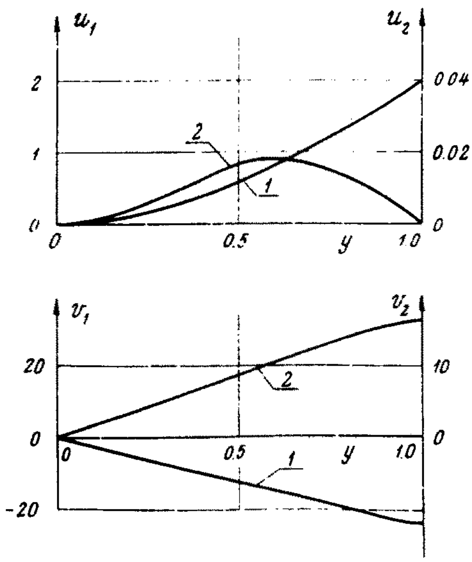

Fig. 3.

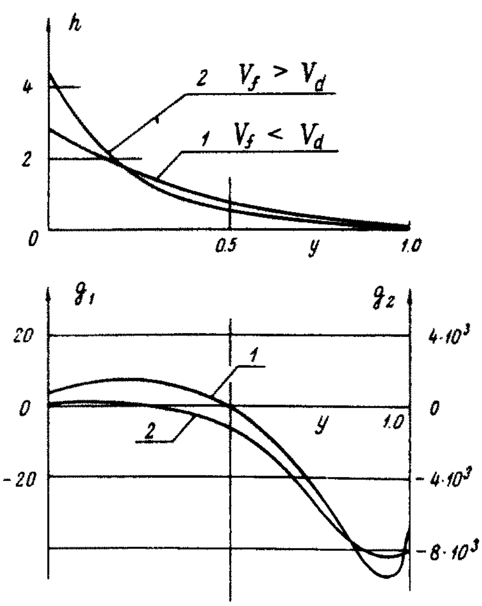

Fig. 4.

with the value $V_f = 52.1$, but note that the divergence speed V_d for this distribution turns out to be smaller, $V_d = 47.8 < V_f$. Hence, the wing with this distribution of material loses its stability by divergence.

It is interesting to note that during the iteration process (3.5), (3.6) from the distribution $h_1(y)$ to $h_2(y)$, the character of the vibrations at the flutter point changes considerably, i.e. some nodal points appear in the eigenfunctions $u(y)$ and $v(y)$. In Fig. 5, the functions $u(y)$ and $v(y)$ corresponding to the distribution $h_2(y)$ are presented. During the iterations the flutter frequency increases from the value $\omega_1 = 224.5$ and up to $\omega_2 = 262.0$.

Thus, we see that the iteration process (3.5), (3.6) of maximizing the critical flutter speed does not lead to a maximization of the smallest of the critical speeds. In order to obtain the true optimal solution it is therefore necessary to use the further iterative formulas (3.5) and (3.8) after attaining the equality $V_f = V_d$. This process converges to the optimal distribution $h^*(y)$ shown in Fig. 6, where it should be noted that $h^*(1) = 0$. The critical speed of aeroelastic instability attains the value $V_f^* = V_d^* = 48.8$, with the frequency of flutter equal to $\omega^* = 265.1$. Thus, the critical speed is increased by 66 pct. relative to the critical value for the uniform distribution $h^{(0)} = 1$.

Note that the estimate of the maximum critical speed (3.3) for the case of constant m_0, GJ_0, b takes the form

$$\max_{h\in\Omega} \min(V_f, V_d) \le V_d^0 = \sqrt{\left(\frac{3GJ_0}{c_m{}^\alpha \rho b^3}\right)} = 65.4.$$

However, the distribution h_d^0 that maximizes the divergence speed V_d^0 is not a solution to our problem (3.1) because it is associated with a lower flutter speed $V_f = 28.9 < V_d^0$.

For the optimal distribution $h^*(y)$, the eigenmodes $u^*(y)$ and $v^*(y)$ of the vibration at flutter are presented in Fig. 7.

We may conclude that the optimal distribution $h^*(y)$ for our problem is characterized by equal critical values of flutter and divergence speeds. Physically this means that the optimal structure will experience two different types of loss of stability—dynamic and static—at the critical point. The similar effect of multiplicity of eigenmodes of optimal structures in self-adjoint problems of elastic stability was discovered by Olhoff and Rasmussen [37] and Medvedev [38], see also [39]. In the self-adjoint case, this effect corresponds to multiplicity of eigenvalues.

The problem of maximizing the critical speed of bending-torsional flutter for given weight has earlier been considered by Vepa [40], but in this work essential errors are made in the

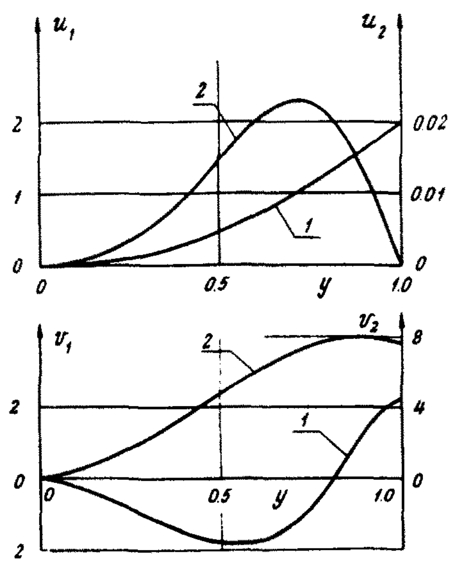

Fig. 5.

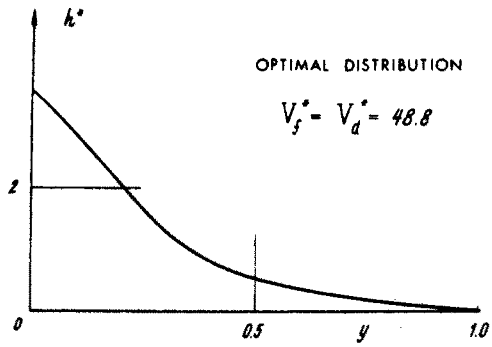

Fig. 6.

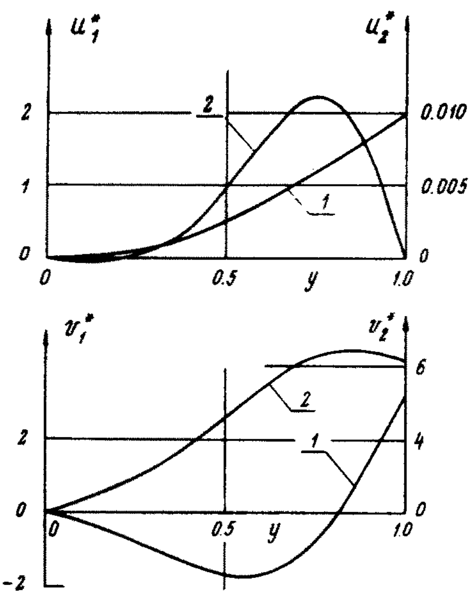

Fig. 7.

derivation of the necessary conditions of optimality: (1) the flutter frequency is not varied and (2) the real and complex quantities are not distinguished. Thus, the conclusions and the numerical results of this work appear to be doubtful.

In the works [6–12] on different problems of optimizing flutter instability systems, direct methods of mathematical programming were employed, i.e. optimality criteria were not used.

4. OPTIMAL ARRANGEMENT OF NONSTRUCTURAL MASS

We now consider the problem of distributing nonstructural mass along the wing span taking into account the influence on the characteristics of aeroelastic stability. Fuel, electronic equipment and some loads may be considered as nonstructural masses.

Let us introduce the control function $h(y)$ by

$$m(y) = m_0(y)(1 + h(y)),\ I_m(y) = I_{m_0}(y)(1 + h(y)) \tag{4.1}$$

where m_0 and I_{m_0} are some fixed distributions of structural mass density and mass moment of inertia per unit length, respectively, while variable quantities $m_0 h$, $I_{m_0} h$ are due to nonstructural masses. It is assumed that adding nonstructural mass does not change the stiffness properties of the wing. However, it will change the inertia properties of the structure and therefore lead to a change of the critical flutter speed. Note, however, that the changes in the inertia properties will have no influence on the critical divergence speed, see (1.8).

In order to calculate the variation of the critical flutter speed due to a variation $\delta h(y)$ of the control function, we consider the main and the adjoint flutter problems (1.5), (1.7), (2.1) and (2.2), and use (4.1). It is easily seen that the first variational calculations are the same as earlier, except that the function H used in the relations (2.6), (2.8) and (2.9) takes the form

$$H = \omega^2 P^T \begin{pmatrix} -m_0 & m_0\sigma \\ m_0\sigma & -I_{m_0} \end{pmatrix} f. \tag{4.2}$$

Thus, the variation δV_f is, as earlier, defined by the formulas

$$\delta V_f = \int_0^1 g\delta h \mathrm{d}y,\ g = -\frac{Im(H\bar{B})}{Im(A\bar{B})} \tag{4.3}$$

where the constants A and B are given by (2.7), and where the function H is now defined by the relation (4.2).

Our optimization problem now consists in determining the distribution $h^*(y)$, which satisfies given constraints and renders the critical flutter speed a maximum possible value

$$\max_{h\in\Omega} V_f(h) = V_f(h^*)$$

$$\Omega = \left\{ h(y): M(h) = \int_0^1 m_0 h \mathrm{d}y = M_0,\ 0 \le h_{\min} \le h(y) \le h_{\max} \right\}. \tag{4.4}$$

Clearly, M_0 denotes the total amount of given nonstructural mass which is to be arranged optimally along the wing span.

By means of the mathematical theory of optimal control [41, 42] it is easily shown that the Hamiltonian for the problem (4.4) is linear in the control function h. This implies that the correct formulation of the optimization problem must include prescribed upper and lower limits $h_{\min}$ and $h_{\max}$, respectively, for the control function because the optimal control will be of the bang-bang type.

Now we shall derive the necessary optimality conditions for the function $h^*(y)$, i.e the solution to the optimization problem (4.4). For this purpose we rewrite the variation δV_f, assuming that the variation δh satisfies the constraints of the problem (4.4), and we add the isoperimetric condition by means of the Lagrangian multiplier μ

$$\delta V_f = \int_0^1 (g + \mu m_0)\delta h \mathrm{d}y.$$

If the control function $h^*(y)$ is optimal then $\delta V_f \le 0$ for admissible variations. Using one-side variations we get

$$\begin{aligned} &g(y) + \mu m_0(y) > 0,\ h^*(y) = h_{\max} \\ &g(y) + \mu m_0(y) < 0,\ h^*(y) = h_{\min}. \end{aligned} \tag{4.5}$$

From these relations we see that the optimal control $h^*(y)$ takes the form

$$h^*(y) = \frac{h_{\max} + h_{\min}}{2} + \frac{h_{\max} - h_{\min}}{2} \operatorname{sgn}(g + \mu m_0) \tag{4.6}$$

where the unknown multiplier μ can be determined from the isoperimetric condition $M(h^*) = M_0$.

As a numerical example, let us again consider a rectangular wing with uniform initial distributions, see [22], wing No. 3. We take the lower limit equal to zero $h_{\min} = 0$. The isoperimetric condition associated with constant m_0 now assumes the following form

$$\int_0^1 h\,dy = \frac{M_0}{m_0} = \frac{\kappa \int_0^1 m_0 dy}{m_0} = \kappa \tag{4.7}$$

where the parameter κ indicates the ratio between the distributed nonstructural mass and the total structural mass of the wing.

In Fig. 8 the gradient of the functional V_f associated with a distribution $h = 0$ (absence of nonstructural mass) is presented. It is immediately seen that in order to increase the critical flutter speed, we have to arrange the mass close to the free tip of the wing, where $g(y) > 0$, see (4.3).

Figures 9 and 10 show optimal distributions $h^*(y)$ and corresponding gradients $g(y)$ for values of (κ, h_{max}) taken equal to (0.45, 4.0) and (0.7, 8.0), respectively. The critical speed V_f associated with these two distributions is equal to 32.1 and 32.9, which exceed by 10–12% the value 29.4, corresponding to absence of nonstructural mass ($h = 0$). The above distributions $h^*(y)$ satisfy the necessary optimality conditions (4.5) with zero value of the multiplier μ.

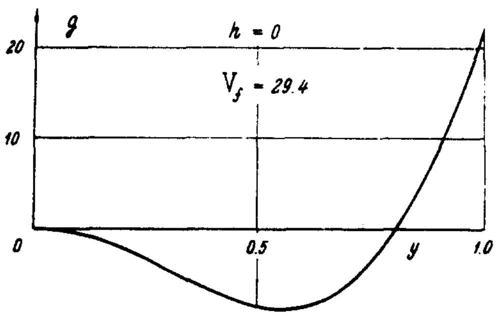

Fig. 8.

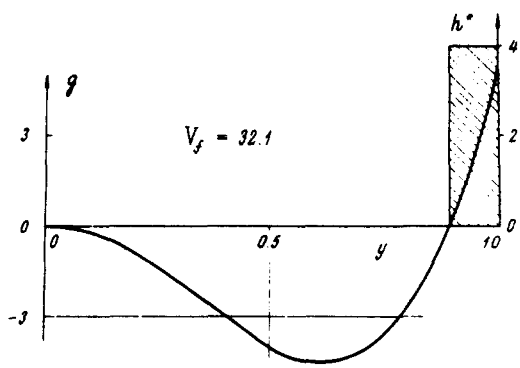

Fig. 9.

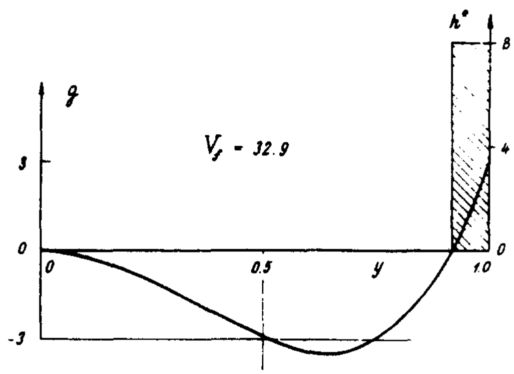

Fig. 10.

For rather large values of the ratio κ/h_{max}, the optimal control function h^* may have two switching points. This case will now be investigated. From the behaviour of the functions $g(y)$ presented in Figs. 8–10, it is natural to expect $h^* = h_{max}$ in the vicinities of $y = 1$ and $y = 0$. Assuming controls with two switching points we obtain

$$h_{max}(y_1 - y_2 + 1) = \kappa$$

by substituting (4.6) into (4.7) and taking $h_{min} = 0$ into account. Here, y_1 and y_2 ($y_2 > y_1$) are the coordinates of switching points. We thus have

$$y_2 = 1 - \frac{\kappa}{h_{max}} + y_1 \tag{4.8}$$

where, due to (4.7), $\kappa/h_{max} \leq 1$.

Thus, we find that the optimization problem with two switching points reduces to maximization of V_f as a function of the single variable y_1, because the second coordinate y_2 will be given by (4.8). Note that the class of controls with two switching points also comprises controls with one switching point.

Figure 11 illustrates optimal distributions $h^*(y)$ with two switching points and corresponding gradient functions $g(y)$ obtained for values of (κ, h_{max}) taken equal to (1, 2) and (2, 4), respectively. The flutter speed V_f attains the values 30.8 and 31.2, which exceed by 5–6% the flutter speed of a wing without added nonstructural mass.

It is interesting to compare optimal mass distributions with some other possible distributions of nonstructural mass. Figure 12 depicts an unfavourable distribution, which is characterized by low value of the flutter speed, $V_f = 23.3$.

On the basis of the results presented in the foregoing, we may draw the conclusion that optimal distributions of nonstructural mass with only one switching point, which appear when the parameter κ/h_{max} is small, are most effective. In such cases the optimal control reduces to a concentrated mass placed at the free tip of the wing. A change in the distribution of the nonstructural mass reduces the flutter frequency significantly (by 30–50% in the cases considered), while the character of the flutter vibrations remains the same.

CONCLUSIONS

In this paper, sensitivity characteristics of aeroelastic stability with respect to changes of distributed and discrete parameters are obtained by means of variational analysis. It is shown

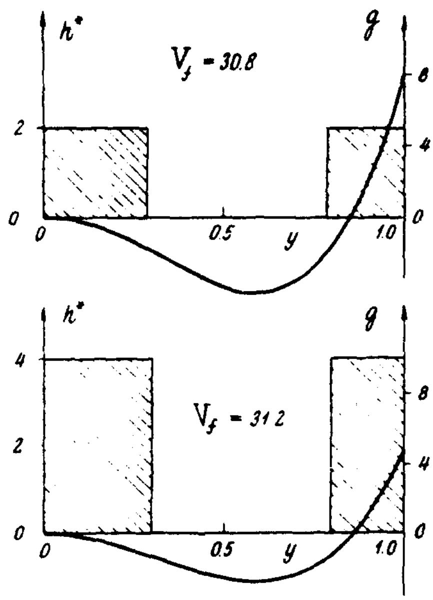

Fig. 11.

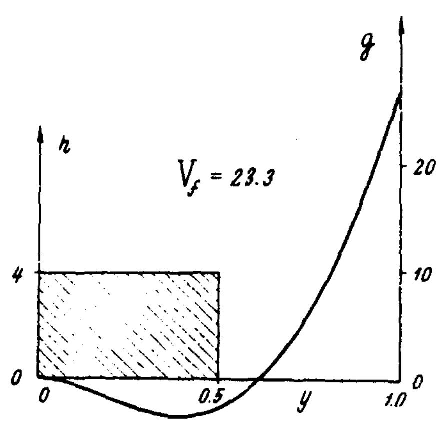

Fig. 12.

that besides the main flutter problem, also the so-called adjoint flutter problem has to be solved in order to calculate the gradient appropriately. Sensitivity analysis offers the designer valuable information by indicating rational ways of improving the structure.

In Section 3, the problem of maximizing the smaller of the flutter and the divergence speed for a given total mass of material is formulated,and it is shown that this problem possesses at least two extrema, but that one of them is only a local maximum. It is also demonstrated that there exist some stiffness and mass distributions for which removal of some structural mass may increase the critical flutter speed, the region near the free tip of the wing being particularly sensitive to variations in the structural mass distribution $h(y)$. Numerical solutions to the maximum optimization problem are given, the interesting feature being that the optimal distribution $h^*(y)$ is characterized by equality of the critical flutter and critical divergence speeds. This implies that the optimal structure experiences both static and dynamic loss of stability at the critical point. This type of behaviour (multiplicity of modes of loss of stability) has earlier been found in optimization problems of elastic stability for conservative systems. For the problem considered in this paper, the critical speed for the optimal wing is increased by 66% in relation to that for a uniform reference wing.

In Section 4 the problem of optimal arrangement of given nonstructural masses along the wing span, is considered. It is shown that the optimal control in this problem is of the bang-bang type. Optimal distributions with one and two switching points are obtained. It is shown that the most effective way of increasing the flutter speed is to locate a concentrated mass at the free tip of the wing.

The sensitivity analysis and the optimization techniques developed in this paper may find application in many other continuous or discrete, nonconservative problems of elastic stability.

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